The Gödel's legacy: revisiting the Logic

Dr. Giuseppe Raguní
UCAM University of Murcia, Spain
graguni@ucam.edu

ABSTRACT: Some common fallacies about fundamental themes of Logic are exposed: the First and Second incompleteness Theorem interpretations, Chaitin's various superficialities and the usual classification of the axiomatic Theories in function of its language order.

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1. Prologue

Regularly, after having enjoyed the fruits of the genius of an extraordinary man, we have to suffer the dogmatism of any his conclusions: everything he said is gold. In physics, an experiment can eliminate the most stubborn opinion, but if the matter is in relation with pure philosophy, things are much more complicated and rarely solvable in a compulsory manner.

However, in some sectors "applied" of philosophy, such as in mathematical Logic, since some time is possible - and imperative - to require accuracy and rigor. Hilbert's formalism introduced by the late nineteenth century, in fact, has equipped the axiomatic Disciplines with a orderly symbolism, deprived of the ambiguity of any intuitive intervention and regimented to a rigorous epistemological analysis. As a consequence, the modern Logic has been capable of remarkable results, led by the astonishing Gödel's theorems.

Despite the above, still today several serious errors are common in this matter. These lapses, combined with the use of a dated terminology which today is indubitably insidious and insufficient, are hindering the spread of this invaluable and fascinating knowledge, and not only in the humanistic field.

Here, I will try to clarify briefly some of these confusions, citing some text where it is possible to find more depth.

2. The pressure of a sentence

The first point concerns the range of applicability of the Gödel's First incompleteness Theorem. We omit, now, both the statement and the very important consequences of this famous theorem, that the reader can find in any good book on

Logic. The thing to emphasize here is just that the same hypothesis of the Theorem require the *enumerability* of the set of theorems and proofs of the Theory to which it can be applied[1]. As a preliminary to the demonstration, Gödel established a special numerical code (called briefly *gödelian*) both for propositions and demonstrations. When applied to the formal[2] Theory of natural numbers (or Peano Arithmetic, *PA* from now), this encoding makes every proposition and every demonstrations be assigned a numeric code, unique and exclusive. But what's happen if you do the same with an arithmetic Theory that has got a number *not enumerable* of theorems? This Theory exists and is usually called "second order Arithmetic". In its premises there is a meta-mathematic axiomatic scheme that, generalizing the principle of induction of *PA*, introduces one axiom for each element of *P(N)*, the set of all the subsets of natural numbers. We will call briefly *complete* this induction. Since this set is, as well known, *not enumerable*, it follows that also the sets of theorems and proofs are not enumerable. Therefore, in this Theory, not all the proofs can have got a different *gödelian*: else its would be enumerable. The not-denumerability of the proofs reveals the indispensable use of a *intrinsically semantic* strategy (ie with use of not fully codifiable meanings) to derive the theorems.

Even if we want to consider the principle of *complete* induction as one symbolic formula, ie as a single semantic axiom, this axiom is *not* decidable (or just effectively enumerable), since its semanticity is not removable. This last conclusion is reached, for example, representing the Theory inside the formal Set Theory[3]: the axiom has to be translated to an axiomatic scheme that generates a number *not enumerable* of inductive axioms[4].

The simple consequence is that the First incompleteness Theorem can not be applied to the second order Arithmetic[5]. However, this ambiguity has been long and loudly spread everywhere, even in specifically technical areas[6]. About this subjet, often is perceived a typical strange lack of rigor (perhaps ill-concealed hint of doubt). For example is repeated usually that "also the second order Arithmetic, since it contains the axioms of *PA,* is subdued by the incompleteness Theorem". Forgetting that also must be kept the *effective axiomatizability*. Emblematic is the case of the Theory obtained from *PA* by adding, as axioms, all the statements true in the *standard model* of *PA*: it also contains all the axioms of *PA,* but it is complete.

Really, it is even possible that the second order Arithmetic is syntactically complete, although its language is semantically incomplete. Since this Theory is not-

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[1] A set is said *enumerable* if it exists a biunivoc correspondence between its elements and the natural numbers.

[2] Throughout the article, with the adjective "formal" we wish to mean "empty of explicit meaning", or "symbolic", "syntactic", "codified".

[3] Anyone between NBG, ZF or MK.

[4] Indeed, a common mistake is to believe that a finite number of statements is always *decidable*. More in: G. Raguní, *Confines lógicos de la Matemática*, revista cultural *La Torre del Virrey - Nexofía*, free on the *Web*: http://www.latorredelvirrey.es/nexofia/pdf/Confines-logicos-de-la-matematica.pdf (2011), pp. 153-158 and 293-295.

[5] In fact in this Theory, the statement of Gödel, whose standard interpretation is "there is no code for a proof of this statement" is not equivalent to "I am not provable in this Theory", as in *PA*.

[6] Just two examples: <<*ovviamente il primo teorema d'incompletezza è dimostrabile anche nell'Aritmetica al secondo ordine*>>, E. Moriconi, *I teoremi di Gödel*, SWIF (2006), on the *Web*: http://lgxserve.ciseca.uniba.it/lei/biblioteca/lr/public /moriconi-1.0.pdf, p. 32; C. Wright, *On Quantifying into Predicate Position: Steps towards a New(tralist) Perspective* (2007), p. 22. In this last work, maybe it is significant that the author discusses this property with a number of delicate epistemologic questions. Anyway, in both cases the property is considered "obvious" without further explanation.

formal, the answer to this interesting question could come only from the Metamathematics[7].

There is no doubt that this situation is due to the overreverence for the Gödel's figure. In the presentation of his incompleteness Theorem in the Congress of Königsberg (1930), Gödel announced the result as "a proof of the semantic incompleteness of Arithmetic, since it is categorical"[8]. He speaks, undoubtedly, of the second order Arithmetic which, unlike *PA*, is categorical[9]. Gödel, therefore, based his claim by applying - wrongly - the First Incompleteness Theorem to the second order Arithmetic: in fact, semantic incompleteness derives from syntactic incompleteness and categoricity. Despite the error, the conclusion is correct: as a result of Löwenheim-Skolem Theorem[10], categoricity and infinity of a model are sufficient to ensure semantic incompleteness of the language of any arbitrary Theory. However, before 1936, when it became popular the generalization of Malcev, no logician, including Skolem and Gödel, realized all the puzzling consequences of this important Theorem.

Now, being the greatest logician of all time, the question is not only what led him to error, but also why his claim has never been subsequently corrected. It is not easy to answer to the first question. It should be noted that Gödel, at least up to 1930, rarely distinguishes the two types of arithmetic Theories, so logically different each from one. Indeed in that period, neither he nor any other logician sometimes highlights the *intrinsic semanticity* of Theories with an uncountable number of statements. And the consequences of it.

Everyone can well understand, on the other hand, why we don't have got any kind of correction about the above announcement. In fact it involves an argument - semantic completeness / incompleteness and its relation with categoricity - that basically, as it was afterward understood, has nothing to do with the syntactic completeness and therefore with the First incompleteness Theorem. During the 30's, this subject was very topical, especially because of Hilbert's concerns about the categoricity. Gödel began showing that the formal classical language[11] is semantically complete: if it is consistent, always has got at least one model[12]. It followed that any formal classical Theory syntactically incomplete, ie, with at least one undecidable statement *I*, could not be categorical. Because it supports at least two nonisomorphic models: one for which *I* is true and another for which is false. In addition, Gödel continued believing - as tacitly assumed by Hilbert and any other logician at time - that the syntactic completeness of a formal Theory (or, more generally, of a Theory with a semantically complete language) should imply its categoricity. Consequently the First incompleteness Theorem was seen, just after its acceptance, as an instrument capable of discriminating the categorical nature and / or the semantic completeness of the axiomatic Systems. For example, the formal arithmetic Theory, ie *PA*, was believed to be not categorical *because* syntactically incomplete.

As noted above, the full understanding of the Löwenheim-Skolem Theorem showed, after a few years, that categoricity is impossible for *all* the formal (or, more generally, with a semantically complete language) Systems equipped with at least one

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[7] G. Raguní, *op. cit.* (footnote n. 4), p. 296-297 .

[8] K. Gödel, *Collected Works I: publications 1929-1936*, eds. S. Feferman et al., Oxford University Press (1986), p. 26-29.

[9] A Theory is said *categorical* if it admits a single model up to isomorphism. In simpler words (but also more inaccurate), if it has got a single correct interpretation.

[10] We include here, both the "up" and "down" version.

[11] We prefer to use this expression rather than the more usual "First Order Classical Logic" for reasons that will become clear in paragraph 4.

[12] Semantic Completeness Theorem, 1929.

infinite model (the case of the ordinary Disciplines). So, this fact is true *regardless if the Theory is syntactically complete or not.* Thus, the subject matter was finally deciphered by consequences of the Completeness Theorem (from which, in fact, the same Löwenheim-Skolem Theorem derives). Neither Gödel, or other alert logician, had ever any good reason to return to a phrase that, ultimately, had diverted, at least, about the consequences of the First incompleteness Theorem. Which certainly were very deep, but concerning essentially different features. The most significant related to the new and disruptive concept of *machine*, as it was elucidated mainly by Church, Turing and subsequently Chaitin.

The sad postscript is that, still today, are frequent claims that, in essence, ratified the Gödel's unfortunate sentence without any correction. For example: <<*the syntactic incompleteness of the first order arithmetic causes the semantics incompleteness of the second order logic*>>[13]. Overlooking the terminology of "expressive order" (that is in effect ambiguous, as we will try to show later), these words seem, firstly, to suggest the automatic transmission of syntactic incompleteness to the expanded System, ie to the second order one (first mistake). Secondly, from syntactic incompleteness and categoricity it is deduced the semantic incompleteness (second lapse: it would sufficient the categoricity plus the infinity of a model).

On the other hand, neither for the language of the integral Theory of reals (at second order, in its original version), informal and categorical System, it is valid the semantic completeness Theorem. Despite that its formal version, expressed at first order, is syntactically complete (as was shown by Tarski). Evidently, the semantic incompleteness of the language of this Theory is just due to its categoricity plus the infinity of the standard model. Here, what role should be played by the syntactic incompleteness of *PA*?

Even in a more recent book, it is proposed to deduce semantic incompleteness of the "second-order logic" either exploiting the First incompleteness Theorem or the Church-Turing Theorem[14]. In the first case the theorem is illegally applied to the second-order Arithmetic. In the second case, the author - agree with others scholars - founds the proof on the fact that <<*if* [by contradiction] *the second-order logic were* [semantically] *complete, then there should be a* [effective] *procedure of enumeration of the valid formulas for the second order...* >>[15]. However, this consequence assumes that the set of the second order formulas is enumerable. But now, if with the expression "second-order logic" we decide to indicate a countable System, then the categorical Arithmetic is not included in it!

Indeed, until syntactic incompleteness (or completeness) of second order Arithmetic remains unproven, we do not distinguish alternatives to use of the Lowenheim-Skolem Theorem to demonstrate the semantic incompleteness of its language.

Finally, it is often repeated also the old slip that the non-categoricity of the formal Arithmetic, ie *PA*, is *caused* by its syntactic incompleteness. That is sort of like saying that the infinity of polygons is *caused* by the infinity of isosceles triangles.

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[13] E. Moriconi, *ibidem* (footnote n. 6), traslated by the author. A similar phrase is repeated in the *abstract* of F. Berto, *Gödel's first theorem*, ed. Tilgher Genova, fasc. Epistemologia 27, n.1 (2004).

[14] C. L. De Florio, *Categoricità e modelli intesi*, ed. Franco Angeli (2007).

[15] C. L. De Florio, *op. cit.*, p. 54, traslated by the author.

3. Chaitin's licenses

In 1970, Gregory Chaitin formulated an interesting informatic version of the First incompleteness Theorem. In its most simple presentation, it states that any machine that verify the Church-Turing Thesis[16] can proof the *randomness* of a necessarily finite number of symbolic strings. The *randomness* of a symbolic string is defined by the impossibility, for the machine itself, to compress it beyond a certain degree (which has to be fixed before). For any machine, exists an infinite and predominant number of random strings: in fact it is easy to conclude that the probability, for a fixed machine, to compress an arbitrary finite string of symbols is always quite low (except to consider a really trivial compression degree). Nonetheless, compressible strings remain certainly infinite; furthermore has to be emphasized that any ordinary human creation, still encoded in symbols, is almost always non-random[17].

Thanks to the Chaitin's interpretation we know that any machine can indicate only a finite number of strings that itself cannot compress. As a result, no compression program, always stopping, can be certainty stated as ideal, ie not improvable: by contradiction, it would be able to determine the randomness of any random string, by virtue of failing to compress it; in violation of this incompleteness Theorem version[18].

Unfortunately Chaitin has released many superficial statements, often uneven, which produce dangerous confusion about the incompleteness subject, already in itself not so easy. The fact that these conclusions are valid also for *universal machines*[19], has led him, first, to neglect that the definition of *randomness* always, in any case, is referred with respect to a particular fixed machine. Moreover, for the obvious fact that, on any ordinary computer, the natural numbers are represented by symbolic strings, he has precipitously assigned the *randomness* property to the natural numbers. As a result of this lightness, Chaitin has repeatedly proclaimed to have discovered the "randomness in Arithmetic"[20].

We reiterate that the *randomness* property (originally defined by A. Kolmorov) only concerns the strings of characters and affects the natural numbers just for the *code* chosen for them. Encoding is totally arbitrary by the point of view of the Logic. In effect it is possible, in a specific machine, to use a code that, although unquestionably uncomfortable and expensive in *bits*, makes *finite* the amount of random natural numbers (or, more exactly, of the strings which represent them)[21]. Furthermore, as before stated, the same randomness of a symbolic string is not absolute, but relative with respect to the code and inner working of the prefixed machine. Given an arbitrary

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[16] This famous "Thesis" is just the assumption that all the machines are logically reproducible using *recursive functions* and vice versa. The *recursive functions* representing all possible arithmetic field of calculability. More accurate and detailed explanations can be found in any good book on Logic.

[17] Hence the success of file compression techniques known as "loss-less", ie without loss or corruption of data, such as "zip", "tar" and so on. However, for products with superior randomness, such as videos and music, compression methods can be effective only with loss or alteration of data: the formats "jpeg" and "mp3" are examples.

[18] G. Raguní, *op. cit.*, p. 273.

[19] A machine is said *universal* if it is able to reproduce logically the behavior of any other machine. Its existence derives, by the Church-Turing Thesis, from the existence of *universal* recursive functions. An example of universal machine is any *PC*, however modest, but with unlimited expandable memory.

[20] Two examples: "*I have shown that there is randomness in the branch of pure mathematics known as number theory. My* work *indicates that - to borrow Einstein's metaphor - God sometimes plays dice with whole numbers!*" *Randomness in Arithmetic*, Scientific American 259, n. 1 (July 1988); "*In a nutshell, Gödel discovered incompleteness, Turing discovered uncomputability, and I discovered randomness*", preface to the book *The unknowable*, ed. Springer-Verlag, Singapore (1999). This kind of frases, however, is repeated in almost all his publications, including the most recent.

[21] Two examples in G. Raguní, *op. cit.*, p. 276 and 281-282.

and long enough string that is random for a particular machine, there is nothing to ban the existence of another machine, possibly designed *ad hoc*, which gets to compress it. For that machine, predictably, some strings that are compressible for the first machines, will be instead random.

In the same article of *Scientific American* cited in the penultimate footnote, Chaitin writes:

> *How have the incompleteness theorem of Gödel, the halting problem of Turing and my own work affected mathematics? The fact is that most mathematicians have shrugged off the results [...]. They dismiss the fact as not applying directly to their work. Unfortunately [...] algorithmic information theory has shown that incompleteness and randomness are natural and pervasive. This suggests to me that the possibility of searching for new axioms applying to the whole numbers should perhaps be taken more seriously.*

> *Indeed, the fact that many mathematical problems have remained unsolved for hundreds and even thousands of years tends to support my contention. Mathematicians steadfastly assume that the failure to solve these problems lies strictly within themselves, but could the fault not lie in the incompleteness of their axioms? For example, the question of whether there are any perfect odd numbers has defied an answer since the time of the ancient Greeks. Could it be that the statement "There are no odd perfect numbers" is unprovable? If it is, perhaps mathematicians had better accept it as an axiom.*

> *This may seem like a ridiculous suggestion to most mathematicians, but to a physicist or a biologist it may not seem so absurd. [...] Actually in a few cases mathematicians have already taken unproved but useful conjectures as a basis for their work.*

These words seem to suggest the "gödelian revolution" of a "new" Mathematics, empirical type, which really has always existed: one which makes use of conjectures. To consider these last as axioms without no metamathematic justification would be obviously unwise as well as useless. And it does not sound like a progress but like a resigned presumption: it desists, a priori, to search for meta-demostrations which often have been precious sources of development for Logic and Mathematics. *Indeed, for no undecidable formula, the incompleteness Theorem impedes the possibility to distinguish it by a purely metamathematic reasoning.* This erroneous view is repeated, with enthusiasm, in almost all his recent work: in fact, on the (epidermal, certainly not logic) basis of the incompleteness Theorem, he goes so far as to question the same opportunity of the axiomatic Systems[22]!

The swedish Logician Torkel Franzén, died recently, exposed other mistakes of Chaitin in 2005[23]. Here, we refer only one. In the abstract of an article, Chaitin states: *<<Gödel's theorem may be demonstrated using arguments having an information-theoretic flavor. In such an approach it is possible to argue that if a theorem contains more information than a given set of axioms, then it is impossible for the theorem to be derived from the axioms>>*[24]. The phrase incorrectly confuses the property that for any machine there are always infinite random strings, with the proving capability of the

[22] G. Chaitin, *The halting probability Ω: irreducible complexity in pure mathematics,* Milan Journal of Mathematics n. 75 (2007), p. 2 y ss.

[23] T. Franzén, *Gödel's Theorem: an incomplete guide to its use and abuse*, AK Peters (2005).

[24] G. Chaitin, *Gödel's Theorem and Information*, International Journal of Theoretical Physics, n. 22 (1982).

machine itself (which is subject to the incompleteness Theorem). Franzén confutes the assertion in a simple and irrefutable way: by the single axiom "$\forall x(x=x)$", having constant complexity, one can obtain a theorem, type "$n = n$", having arbitrarily large complexity by increasing the number n. Indeed, that is guaranteed if the natural numbers are encoded by any usual exponential code.

The constant Ω, introduced by Chaitin with respect to a fixed universal machine, represents the probability that a random routine of the machine stops. Its unquestionable interest lies in the fact that it represents a kind of *best compression* of mathematical knowledge: knowledge of the first *n bits* of Ω can solve the halting problem for all programs of length less than or equal to *n*. However, Chaitin greatly exaggerates its importance, as far as to describe Ω as *the way* to obtain the mathematical knowledge[25]. Naturally, the number Ω is just a fantastic, insuperable way to *summarize* this knowledge. After having obtained it by the traditional theorems and meta-theorems.

This criticism is not meant to attack the figure of this great informatic logician, but just to clarify a picture that shows still quite confused, not only to non-experts.

4. A tired classification

We now wish to criticize the current classification of the classical axiomatic Theories in function of its expressive order: *first order*, *second order*, etc..[26]. Unfortunately, it has been consolidated over the years the view that the formality of language (or, more generally, its semantic completeness) is a prerogative of the first expressive order and, moreover, that higher-order languages are all necessarily semantic (typically, uncountable). The error is mainly due to two misunderstandings.

The first is linked to the meaning of *First Order Classical Logic*. This expression usually refers to the collection of all the *Classic Predictive Calculi*. Each *Classical Predictive Calculus* is a first order formal Theory that has got: a) some basic first order axioms, specified for the first time by Russell and Whitehead, which formalize the concepts of "not", "or" and "exists"[27]; b) some other own axioms, again enumerable and at first order, which formalize some particular concepts that have to be used in the Theory (eg, "equal", "greater than", "orthogonal", etc.); c) the four classical deductive rules: *substitution, particularization, generalization* and *modus ponens*. Since these rules consist of purely *syntactic* operations on axioms and/or theorems (ie are applied, mechanically, just to symbolic code of the statements), in any Predictive Calculus - and so in the whole *Classical First Order Logic* - the formality is always verified. However, this fact does not mean that *every* first order Theory, founded on a particular Classical Predictive Calculus and deducting only by the four classic rules, must be formal. Nothing prevents, for example, that a Theory add an uncountable number of own axioms to this Calculus, although all expressed at first-order language: it would result an intrinsically semantic and therefore not formal language. The *First Order Classical*

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[25] For example, in the article: *Meta-mathematics and the foundations of Mathematics*, EATCS Bulletin, vol. 77, pp. 167-179 (2002), he exposes a method that in principle should be able to solve the Riemann hypothesis from knowing a sufficient number of bits of Ω. But the argument is a clear vicious circle.

[26] A language is said of the first-order if the quantifiers "$\exists$" and "$\forall$" only may refer to variables (as in the phrase "*every* line that is parallel to line *r* also is parallel to line *t*"). At second order, also it can be quantified on the predicates (and to translate phrases like: "*Every* relationship that exists between the lines *r* and *s* also exists between the lines *r* and *t*"). At third order it can be quantified also on *super-predicates* (relations between predicates) and so on to infinity.

[27] Other usual classical concepts, such as "and", "imply" and "every", are defined by them.

Logic, in other words, does not include all the classical first order Theories. Infinitely many of them use an informal and/or semantically incomplete language[28].

The second mistake is related to the Lindström's Theorem. This states that any classical Theory expressed in a semantically complete language *can* be formulated by a first order language. The disorder stems to confuse "can" with "must". *The theorem does not forbid the semantic completeness, or formality, of higher-order languages*. It only states that when you have just this case, the Theory can be re-expressed more easily at first order language. Unquestionably, this property distinguishes the particular importance of the first order expressions. A property, on the other hand, that already is evident thanks to the expressive capability of the formal Set Theory: any *formal* System, in fact, since is fully representable inside this Theory - which is of first order - is expressible at first order. But this importance should not be radicalized.

Surely, the evident fact that the second order languages are typically uncountable, and therefore intrinsically semantic, aggravates the situation. That happens because, if the model is infinite, in the most general case the predicates vary within a certainly not enumerable set. But nothing preclude that the axioms limit this variability to a countable subset and that, in particular, the formality is respected[29]. A concrete example is the System obtained from *PA,* expressing the *partial* induction principle, ie limited to formulas with at least one free variable, by a single symbolic formula (instead of by a meta-mathematic axiomatic scheme). It results a second-order axiom which still has to be interpreted semantically, since the premises of the Theory do not specify any syntactic deduction with second order formulas. However, this semantics is not *intrinsic*, ie ineliminable: if we represent the System inside the formal Set Theory, this axiom is translated to a set-theoretic axiomatic scheme that generates an *enumerable* number of formal inductive axioms. Therefore, in the same original Theory, the formality could be restored by adding the appropriate premises with which syntactically operate from the second order induction axiom, so as to produce the required theorems about the symbolically declarable properties. But, naturally, there are strategies more simple to reconstitute the formality[30].

As a result of this confusion, the non-formal nature of the second (or more) order Theories is often criticized, on the base of intrinsic semanticity in *some* of these. And others scholars, rather than highlight that the problem does not lie in the kind of expressive order, but in the particular nature of the premises of the Theory, contest that "also some second order Systems have a formal appearance like those of the first order". In any case, like *some* of the first order.

In conclusion, the cataloging of classical axiomatic Theories in function of its language order, in general, misleads about their basic logical properties. Those are consequence of the structure of the premises, whereas the language order not always plays a decisive role. The main instrument of classification remains the respect of the Hilbert's formality or, more generally, of the semantic completeness.

5. About the interpretation of the Second incompleteness Theorem

[28] In agree: M. Rossberg, *First-Order Logic, Second-Order Logic, and Completeness*, Hendricks et al. (eds.) Logos Verlag Berlin (2004), on the *Web*:
http://www.st-andrews.ac.uk/~mr30/papers/RossbergCompleteness.pdf
[29] In agree: HB Enderton, *Second order and Higher order Logic*, Standford Encyclopedia of Philosophy (2007), on the *Web*: http://plato.standford.edu/entries/logic-higher-order/.
[30] G. Raguní, *op. cit.*, p. 160-161.

Finally, we have to disrupt the usual interpretation of the Second incompleteness Theorem. In reference to a Theory that satisfies the same hypothesis of the First incompleteness Theorem, the Second one generalizes the undecidability to a class of statements which, interpreted in the *standard model* mean "this System is consistent." Its complex demonstration, only outlined by Gödel, was published by Hilbert and Bernays in 1939.

The usual interpretation of this Theorem, object of our criticism, is that "every Theory that satisfies the hypotheses of the First incompleteness Theorem can not prove its own consistency." It seems clear, in fact, that the conclusion that a Theory can not prove its own consistency is valid for *all the classical Systems*, including non-formal. Moreover, this conclusion does not correspond to the Second incompleteness Theorem, but to a new metatheorem.

Consider an arbitrary classical System. If it is inconsistent, it is deprived of models and therefore of any reasonable interpretation of any statement[31]. Therefore, only the admitting that a given statement of the Theory means something, implies agreeing consistency. And indubitably this also applies if the interpretation of the statement is "this System is consistent".

So, if there is no assurance about the consistency of the Theory (which, to want to dig deep enough, applies to any mathematical Discipline) we can't be certain on any interpretation of its language. For example, in the case of the usual Geometry, when we prove the pythagorean Theorem, what we really conclude is "if the System supports the Euclidean model (and therefore is consistent), then in every rectangle triangle $c_1^2 + c_2^2 = I^2$". Certainly, a deduction with undeniable epistemological worth, still in the catastrophic possibility that, in a sad day, the System were revealed as inconsistent.

But now let's see what's happen if a certain theorem of a certain Theory is interpreted with the meaning: "this System is consistent" in a given interpretation M. Similarly, what we can conclude by this theorem is really: "if the System supports the model M (and therefore is consistent), then the System is consistent." *Something that we already knew and, above all, that does not demonstrate at all the consistency of the System.* Unlike any other statement with a different meaning in M, for this kind of statement we have a peculiar situation: *bothering to prove it within the Theory is epistemologically irrelevant in the ambit of the interpretation M.* In more simple words, the statement in question can be a theorem or be undecidable with no difference for the epistemological view. Just it cannot be the denial of a theorem, if M is really a model. So, in any case, *the problem of deducing the consistency of the Theory is beyond the reach of the Theory itself.* We propose to call *Metatheorem of undemonstrability internal of consistency* this totally general metamathematic conclusion.

Then, the fact that in a particular hypothetically consistent Theory, such a statement is a theorem or is undecidable, is depending on the System and on the specific form of the statement. For Theories that satisfy the assumptions of the First, Second incompleteness Theorem guarantees that "normally" these statements are undecidable. We say "normally" because apparently there are also statements, still expressing consistency of the System in other peculiar models that, on the contrary, turn out to be *theorems* of the Theory[32]. As the same Lolli says, "it seems that not even a proof shuts

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[31] More in depth: of any interpretation respecting the principles of contradiction and excluded third.
[32] See eg: G. Lolli, *Da Euclide a Gödel*, ed. Il Mulino (2004), p. 140 y 142 and A. Martini, *Notazioni ordinali e progressioni transfinite di teorie*, Thesis Degree, University of Pisa (2006), p. 11-15, on the *Web*: http://etd.adm.unipi.it/theses/available/etd-11082006-161824/unrestricted/tesi.pdf

discussions"[33]. But in any case, as before concluded, this debate cannot affect the validity of the proposed Metatheorem of undemonstrability internal of consistency.

In conclusion, the Second incompleteness Theorem identifies another class of essentially undecidable statements for any Theory that satisfies the hypothesis of the First one. Whilst the First incompleteness Theorem determines only the Gödel's statement, the Second extends the undecidability to a much broader category of propositions. But, contrary to what is commonly believed, this drastic generalization does not introduce any new and dramatic epistemological concept about the consistency of the System. It doesn't, even if the Theorem were valid *for every* statement interpretable as "this System is consistent" (which, we reaffirm, seems to be false). Because by it, in any case, we cannot conclude that "the System cannot prove its own consistency": this judgment belongs to a completely general metatheorem which seems never have been stated, despite its obviousness and undeniability[34].

Often the misreading is aggravated for a sort of incorrect "intuitive proof" of the Second incompleteness Theorem; that sounds so: "Let *S* be a System that satisfies the hypotheses of the First incompleteness Theorem and *C* its statement affirming the consistency of *S*. The First incompleteness Theorem shows that if *S* is consistent, the Gödel's statement, *G*, is undecidable. Now, if *C* were provable, we could deduce that *G* is undecidable and therefore unprovable. But since *G* claims to be itself unprovable, this fact would mean just to prove *G*, that is absurd. Therefore, *C* must be unprovable"[35]. The flaw is obvious: the reasoning gives to *C* and *G* a semantic value which is certified only assuming that the System admits a model with such interpretations and therefore already assuming that it is consistent. *In this model* there is no doubt that the truth of *C* implies the truth of *G*, but the syntactic implication $C \rightarrow G$ is a totally different question. In general, the possibility that *C* is a theorem causes no absurd consequence, because that does not imply really the consistency of the System, as it was observed. On the other hand, in case of inconsistency, maybe doesn't happen that every statement is a theorem?

Actually, the syntactic inference $C \rightarrow G$ is not so trivial and, as noted, it does not apply in all cases but depends on the symbolic structure of the statement *C*.

The only epistemological result of effective importance about the consistency is due to the Metatheorem of undemonstrability internal of consistency. And we emphasize that its meta-demonstration, since refers to any arbitrary classical System, must consist in a purely meta-mathematical reasoning (like that one we did), ie it cannot be formalized.

In the book, already mentioned, *Confines Lógicos de la Matemática*[36], we propose other revisions and some proposals to update the terminology of these arguments, which continues unchanged since the time of Hilbert.

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[33] G. Lolli, *op. cit.*, p. 142.

[34] The consistency of a System may be demonstrated only outside the same, by another external System. For which, in turn, the problem of consistency arises again. To get out of this endless chain, the "last" conclusion of consistency has to be purely metamathematic. Actually, the most general Theory (that demonstrates the consistency of the ordinary mathematic Disciplines) is the formal Set Theory and the conclusion of its consistency only consists in a "reasonable conviction".

[35] See eg: P. Odifreddi, *Metamorfosi di un Teorema*, (1994), on the *Web*:
http://www.vialattea.net/odifreddi/godel.htm

[36] Also in italian: *I confini logici della matematica*, on ed. Aracne, Bubok, Scribd, Lulu or Amazon.